ANSWER KEY

PERTH COLLEGE
YR 12 3CD SPECIALIST MATHEMATICS
SEMESTER ONE 2010
TEST 1
VECTORS (70%) DIFFERENTIATION (30% - functions 22%, trig 8%)
Simple (~70%) Complex (~30%)

Name: ______________________________

Time Allowed: 60 minutes / 50 = %

SECTION ONE	**/20**
SECTION TWO	**/30**

SECTION ONE: CALCULATOR FREE

TIME: 25 minutes **TOTAL MARKS: 20**

- Answer all questions neatly in the spaces provided.
- **Show all working** where appropriate.
- **One side** of an A4 sheet of paper for notes is allowed for Section 2 only.
- Formula sheet may be used for both sections.

Question 1 (2, 1, 2 = 5 marks)

Given $m = 4\boldsymbol{i} + 6\boldsymbol{j} + \boldsymbol{k}$, $p = 2i - 3j$ *and* $n = 8\boldsymbol{i} + 4\boldsymbol{j} + 2\boldsymbol{k}$

a) Find $2p - 4n$

$$2\begin{pmatrix}2\\-3\\0\end{pmatrix} - 4\begin{pmatrix}8\\4\\2\end{pmatrix} = \begin{pmatrix}4\\-6\\0\end{pmatrix} - \begin{pmatrix}32\\16\\8\end{pmatrix} \checkmark = \begin{pmatrix}-28\\-22\\-8\end{pmatrix} \checkmark$$

b) Find $|m|$

$$|m| = \sqrt{4^2+6^2+1^2} = \sqrt{53} \text{ units} \checkmark$$

c) Find $n \cdot p$

$$\begin{pmatrix}8\\4\\2\end{pmatrix} \cdot \begin{pmatrix}2\\-3\\0\end{pmatrix} = 8.2 + 4.(-3) \checkmark = 4 \checkmark$$

/5

Question 2 (1, 1, 1, 2, 1, 1 = 7 marks)

Three points in space are given:

$$P(2,2,0) \qquad Q(1,1,1) \qquad R(2,-1,2)$$

Find:

a) $\overrightarrow{PQ}$ $= -P + Q$

$$\begin{pmatrix} -2 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \checkmark$$

b) $|\overrightarrow{PQ}|$

$$\sqrt{1^2 + (-1)^2 + (-1)^2} = \sqrt{3} \text{ units} \checkmark$$

c) The parametric equations of the line through P and Q.

$$\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \therefore \left.\begin{matrix} x = 2 - \lambda \\ y = 2 - \lambda \\ z = \lambda \end{matrix}\right\} \checkmark$$

No 1/2 marks

d) The Vector equation of the plane through P, Q, and R in the form:

(i) $ax + by + cz = d$ given $d = 6$

$$ax + by + cz = 6$$

$$\left\{\begin{matrix} 2a + 2b = 6 \\ a + b + c = 6 \\ 2a - b + 2c = 6 \end{matrix}\right\} \checkmark$$

Solve eqn 2 + 3

$$\therefore 3b = 6$$
$$b = 2$$
$$a = 1$$
$$c = 3$$
$$\therefore x + 2y + 3z = 6 \checkmark$$

(ii) $r.n = c$

$$r \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 6 \checkmark$$

(iii) $r = a + \lambda b + \mu c$

$$r = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \checkmark$$

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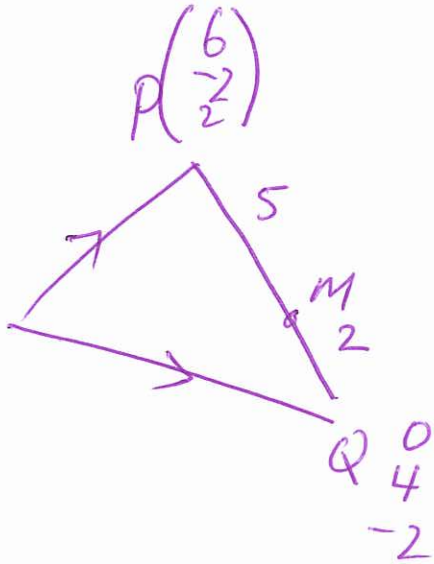
Question 3 (2, 1 = 3 marks)

a) Two points in space are given;

$$P(6,-2,2) \qquad \textit{and} \qquad Q(0,4,-2)$$

Find the coordinates of M which divides QP in the ratio 2:5



$$\overrightarrow{QP} = P - Q = \begin{pmatrix} 6 \\ -6 \\ 4 \end{pmatrix}$$

$$M = Q + \frac{2}{7} QP$$

$$M = \begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix} + \frac{2}{7}\begin{pmatrix} 6 \\ -6 \\ 4 \end{pmatrix}$$

$$M = \begin{pmatrix} 12/7 \\ 17/7 \\ -6/7 \end{pmatrix}$$

b) Find a unit vector parallel to $\begin{pmatrix} -8 \\ 0 \\ 6 \end{pmatrix}$.

$$\sqrt{(-8)^2 + 6^2} = 10$$

$$\therefore \frac{1}{10}\begin{pmatrix} -8 \\ 0 \\ 6 \end{pmatrix}$$

OR $\dfrac{-1}{10}\begin{pmatrix} -8 \\ 0 \\ 6 \end{pmatrix}$

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Question 4 (1, 1, 1, 2 = 5 marks)

Given the following functions find $^{dy}/_{dx}$:
(Answer in positive indices)

a) $y = x^4(3x + 4x^3 + 2)$

$$\frac{dy}{dx} = 4x^3(3x + 4x^3 + 2) + x^4(3 + 12x^2) \checkmark$$

b) $y = 2\cos(\cos x)$

$$\frac{dy}{dx} = 2\sin(\cos x)\sin x \checkmark$$

c) $y = \sin u \quad$ *and* $\quad u = x^2 - 3x$

$$\frac{dy}{du} = \cos u \qquad \frac{du}{dx} = 2x - 3$$

$$\frac{dy}{dx} = (2x-3)\cos(x^2 - 3x) \checkmark$$

d) $y = \frac{\sqrt{x}+4}{\sqrt{x}-4}$

$$\frac{dy}{dx} = \frac{\frac{1}{2}x^{-1/2}(x^{1/2} - 4) - \frac{1}{2}x^{-1/2}(x^{1/2} + 4)}{(\sqrt{x} - 4)^2} \checkmark$$

$$= \frac{\frac{1}{2} - 2x^{-1/2} - \frac{1}{2} - 2x^{-1/2}}{(\sqrt{x} - 4)^2}$$

$$= \frac{-4}{\sqrt{x}(\sqrt{x} - 4)^2}$$

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SEMESTER ONE 2010
TEST 1
VECTORS (60%) & DIFFERENTIATION (40%)

Name: ____________________

SECTION TWO: CALCULATOR ALLOWED

TIME: 35 minutes **TOTAL MARKS: 30**

- Answer all questions neatly in the spaces provided.
- **Show all working** where appropriate.

Question 5 (3 marks)

Find the distance between the parallel planes $r \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 10$ *and* $r \cdot \begin{pmatrix} 9 \\ 6 \\ 3 \end{pmatrix} = 22.$

(Show clear justification to gain full marks).

ANY point on plane $r \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 10$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 0$

$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$ ✓

P.2 $9x + 6y + 3z = 22$

$$\frac{(9)(1) + (6)(1) + (5)(3) - 22}{\sqrt{9^2 + 6^2 + 3^2}}$$ ✓

$= 8/\sqrt{126}$ unit ✓

/3

Question 6 (1, 1, 2, 2 = 6 marks)

Line L has a vector equation $r = 4\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k} + \lambda(2\boldsymbol{i} + 3\boldsymbol{j} - 4\boldsymbol{k})$

Plane P_1 has a vector equation $r \cdot (3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}) = 5$

Plane P_2 has a vector equation $r \cdot (-2\boldsymbol{i} + a\boldsymbol{j} + 2\boldsymbol{k}) = -8$

a) Clearly show whether the point A, with position vector $(8\boldsymbol{i} + 5\boldsymbol{j} - 6\boldsymbol{k})$, lies on the Line L.

$4 + 2\lambda = 8 \quad \lambda = 2$

$-1 + 3\lambda = 5 \quad \lambda = 2$

$2 - 4\lambda = -6 \quad \lambda = 2$

$\therefore$ same line ✓

b) Clearly show whether the point B, with position vector $(3\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k})$, lies on the plane P_1.

$(3)(3) + (3)(-2) + (2)(1) = 5$ ✓

$r \cdot n = 5 \quad \therefore$ lies on P_1

c) Clearly show that Line L does not lie in Plane P_1.

$3(4+2\lambda) + (-2)(-1+3\lambda) + (1)(2-4\lambda) = 5$ ✓

$14 - 4\lambda = 5$

Does not work for all values of λ $\therefore$ Does not lie on line ✓

d) Determine the value of "a", such that Line L lies in the plane P_2.

$(-2)(4+2\lambda) + (a)(-1+3\lambda) + (2)(2-4\lambda) = -8$ ✓

$-8 - 4\lambda - a + 3a\lambda + 4 - 8\lambda = -8$

$a = 4$ for all $\mathbb{R}$ values of λ ✓

6

Question 7 (1, 1, 2 = 4 marks)

If $\boldsymbol{a} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}$ *and* $\boldsymbol{b} = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}$ find;

a) The angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ to the nearest degree.

$$a.b = |a|\,|b| \cos\theta$$

$$18 = \sqrt{17}.\ \sqrt{24} \cos\theta$$

$$\theta = 27° \checkmark$$

b) The vector equation of the line passing through point A, position vector $\boldsymbol{a}$, and point B, position vector $\boldsymbol{b}$.

$$\overrightarrow{ab} = -a + b = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

$$r = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \checkmark$$

c) The vector equation of the plane containing point A and perpendicular to line AB. *(To gain full marks you must fully justify your solution).*

$$r \cdot \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \checkmark$$

$$r \cdot \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} = 1 \checkmark$$

$/4$

Question 8 (3, 1, 1 = 5 marks)

a) **Show** that the equation of the tangent to $y = tan\ (x)$ at the point where $x = \frac{\pi}{4}$ is $y = 2x + (\frac{2-\pi}{2})$

at $\pi/4$ $y = 1$ ✓

$y = \tan(x)$

$\frac{dy}{dx} = \frac{1}{\cos^2(x)}$

at $\pi/4$

$dy/dx = 2 \therefore m = 2$ ✓

$y = 2x + c$

$1 = 2(\pi/4) + c$

$\therefore c = 1 - \pi/2$

$\therefore y = 2x + (1 - \pi/2)$ ✓

b) If $f(x) = 4 \sin^5(x)$ find: *(to 3 DP)*

(i) $f'(\frac{\pi}{3})$

$5\, 5/8$ ✓

Calculator is fine.

(ii) $f''(\frac{\pi}{3})$

3.248 ✓

/5

Question 9 (3, 2 = 5 marks)

a) Find the points on the curve $xy - y - x = 1$ where the tangent is parallel to the line $x + 4y = 1$.
(To gain full marks you must fully justify your solution).

$xy - y - x = 1$

$y(x-1) = 1 + x$

$y = \frac{1+x}{x-1}$

$\frac{dy}{dx} = \frac{(1)(x-1) - (1)(1+x)}{(x-1)^2}$

$= \frac{x - 1 - 1 - x}{(x-1)^2}$

$= \frac{-2}{(x-1)^2}$ ✓

$x + 4y = 1$

$m = -1/4$

$\therefore -\frac{1}{4} = \frac{-2}{(x-1)^2}$

$\therefore -(x-1)^2 = -8$

$(x-1)^2 = 8$

$x = 3.8284$
$y = -0.7071$ ✓

$x = -1.8284$
$y = 0.7071$ ✓

b) Indentify the function being differentiated and hence find its exact value:

$$\lim_{h \to 0} \left[\frac{\tan(\frac{\pi}{3} + h) - \sqrt{3}}{h} \right]$$

$\lim_{h \to 0} \frac{\tan(\pi/3 + h) - \sqrt{3}}{h}$

$y = \tan(x)$ at $\pi/3$ ✓

$\frac{dy}{dx} = \frac{1}{\cos^2(\frac{\pi}{3})} = 4$ ✓

/5

Question 10 (7 marks)

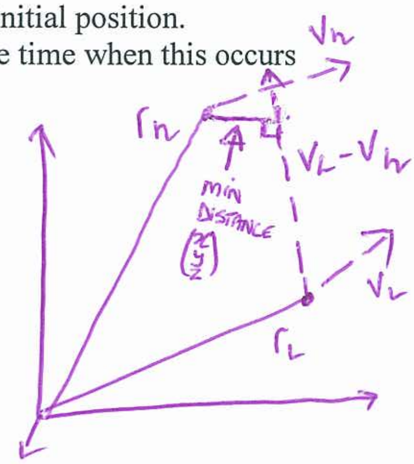
Luke and Han are flying their B wing aircraft. The initial position vectors relative to the home base (the origin) are:

$$r_{Luke} = \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} \textit{ kilometres} \quad \textit{and} \quad r_{Han} = \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix} \textit{ kilometres.}$$

They start flying at the same time with velocities:

$$v_{Luke} = \begin{pmatrix} -2 \\ -1 \\ 3 \end{pmatrix} \textit{ kilometres per hour} \quad \textit{and} \quad v_{Han} = \begin{pmatrix} 1 \\ -2 \\ 7 \end{pmatrix} \textit{ kilometres per hour.}$$

If they collide, state the time of the collision after the aircraft left their initial position.
If they do not collide, state the minimum distance they are apart and the time when this occurs (after the aircraft leave their initial position).



MIN DISTANCE (d) $= \begin{pmatrix} 4-2t \\ -t \\ 1+3t \end{pmatrix} - \begin{pmatrix} -4+t \\ 1-2t \\ 3+7t \end{pmatrix}$

$(d) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 8-3t \\ -1+t \\ -2-4t \end{pmatrix}$ ✓

$\therefore |d| = \sqrt{(8-3t)^2 + (-1+t)^2 + (-2-4t)^2}$

$= \sqrt{69-34t+26t^2}$ ✓

for $d=0$ there is No $\mathbb{R}$ solution ..

$\therefore$ no collision ✓

$d' = \dfrac{-34+52t}{2(69-34t+26t^2)^{1/2}}$

Can be done without calculus

MIN occurs at $d'=0$

$\therefore 0 = -66+34t$ ✓

$t = 0.6538$ sec ✓

$\therefore d = 7.6082$ km ✓✓

$\frac{}{7}$